Generalized quantization scheme for Lie algebras

1992 J. Phys. A: Math. Gen. 25 L1139

(http://iopscience.iop.org/0305-4470/25/19/001)

J. Phys. A: Math. Gen. 25 (1992) L1139–L1143. Printed in the UK

LETTER TO THE EDITOR

Generalized quantization scheme for Lie algebras

V Lyakhovsky†§ and A Mudrov‡

† Institut für Theoretische Physik, Freie Universität Berlin, W-1000 Berlin 33, Federal Republic of Germany

‡ Institute of Physics, St Petersburg State University, St Petersburg 198904, Russia

Received 5 May 1992

Abstract. The generalized form of the coproduct for quantum Lie algebras is proposed. It is demonstrated that the admissible quantum compositions can be easily found. The whole construction gives rise to the wide class of multidimensional Hopf algebras. The efficiency of the method is demonstrated on examples of non-semisimple quantum universal enveloping algebras.

Investigating the structure of the quantized simple Lie algebras relations proposed by Drinfeld and Jimbo [1, 2] and their R-matrix realization discovered by Faddeev, Kulish, Reshetikhin and Takhtajan [3, 4], having in mind the multiparametric quantizations [5] and the contractions as a tool for building the quantum non-semisimple Lie algebras [6–8], one necessarily comes to the following conclusions. (i) The Drinfeld–Jimbo method of quantization of universal enveloping algebras must have multidimensional generalizations. (ii) The universal enveloping algebras $U(\mathscr{G})$ for non-semisimple $\mathscr{G}$ must have the direct method of their quantum deformations.

In this letter we describe such a path of quantization that can help to solve these problems. We propose the explicit form of the coproduct for the wide class of Hopf algebras so that the determination of the admissible quantum compositions is simplified. The basic structure is formulated in the proposition. Its role in the quantization is illustrated by the explicit construction of non-semisimple quantum Lie algebras—the non-trivial multidimensional deformations of $U(\mathscr{G})$.

We shall use the following notations:

$\mathscr{A}$ – the associative algebra over the field $\boldsymbol{K}$;

H_i, X_l – the generators of $\mathscr{A}$;

I, L – the finite sets with volumes $|\mathsf{I}|$ and $|\mathsf{L}|$;

$i, j, k \in \mathsf{I}$; $l, m, n, \in \mathsf{L}$;

α^i, β^j – the commuting $|\mathsf{L}| \times |\mathsf{L}|$ matrices;

$X'_m \equiv X_m \otimes 1$, $X''_m \equiv 1 \otimes X_m$, for $m \in \mathsf{L}$;

X – the vector whose components are the generators X_m;

$H' \cdot \alpha'' \equiv \Sigma\, H_i \otimes \alpha^i$ – the operator, transforming vector $v \otimes w \in \mathscr{A} \otimes \mathscr{A}$ into $\Sigma\, H_i v \otimes \alpha^i w$. Here vector v is multiplied by H_i and matrix α^i transforms the subspace spanned by X_m and acts trivially elsewhere;

$\beta' \cdot H''$ is defined similarly;

$\vec{H} \cdot \alpha \equiv \Sigma\, \vec{H}_i \cdot \alpha^i$ transforms vector $v \in \mathscr{A}$ to $\Sigma\, H_i \alpha^i v$ and similarly

$\beta \cdot \tilde{H} \equiv \Sigma\, \beta^i \cdot \tilde{H}_i : v \to \Sigma\, \beta^i v H_i$.

§ Permanent address: Department of Theoretical Physics, Institute of Physics, St Petersburg State University, St Petersburg 198904, Russia.

0305-4470/92/191139+05$04.50

Proposition. The free associative unital algebra $\mathscr{A}$ over the field $\boldsymbol{K}$, generated by $1, H_i, X_m$ ($i \in \mathsf{I}$, $m \in \mathsf{L}$) with the coproduct

$$\Delta X = \mathrm{e}^{H'\cdot\alpha''} X'' + \mathrm{e}^{\beta'\cdot H''} X' \tag{1}$$

$$\Delta H_i = H_i' + H_i'' \tag{2}$$

$$\Delta 1 = 1 \otimes 1 \tag{3}$$

the counit

$$\varepsilon(H_i) = \varepsilon(X_m) = 0 \qquad \varepsilon(1) = 1 \tag{4}$$

and the antipode

$$S(H_i) = -H_i \tag{5}$$

$$S(X) = -\mathrm{e}^{-(\beta\cdot\vec{H})}\,\mathrm{e}^{-(\vec{H}\cdot\alpha)} X \tag{6}$$

$$S(1) = 1 \tag{7}$$

is the Hopf algebra if the generators H_i commute.

The statement is easily checked by the direct verification of the Hopf algebra axioms [9]. When the matrix elements of α^i and β^i tend to zero, formulae (1)–(7) obtain the characteristic form of the universal enveloping algebra. So they can be regarded as describing the quantized version of some $\mathrm{U}(\mathscr{G})$.

Let $\mathscr{G}$ be the Lie algebra generated by H_i and X_m where $\{H_i\}$ form the basis of an Abelian subalgebra. Write down the deformation of the product in $\mathrm{U}(\mathscr{G})$:

$$[g_r, g_s] = [g_r, g_s]_0 + \Phi_{rs}(\alpha^i, \beta^j; H_k). \tag{8}$$

Here $[g_r, g_s]_0$ is the initial composition and the deforming functions Φ_{rs} depend on α^i and β^j and are the power series of H_k. Define the coproduct (1)–(3) and impose the condition that this coproduct is an algebraic map. The explicit form of coproduct simplifies considerably the determination of Φ_{rs}. To end the procedure the Jacobi identity must be checked.

Thus the proposition can be used to construct the family of quantum universal enveloping algebras $\mathrm{U}_{\alpha,\beta}(\mathscr{G})$. In this scheme of quantization we are not limited to dealing only with simple Lie algebras. Here it is not even necessary to use the contraction procedure. The wide class of non-semisimple quantum Lie algebras can be obtained directly.

To illustrate these properties we give the examples of quantum $\mathrm{U}_{\alpha,\beta}(\mathscr{G})$ algebras for three-dimensional $\mathscr{G}$s.

Example 1. Consider the Heisenberg algebra $h(1)$ with the generators H, X_1, X_2.

$$[H, X_1] = [H, X_2] = 0 \tag{9}$$

$$[X_1, X_2] = H. \tag{10}$$

Let α and β be commuting complex 2×2 matrices. Then according to (1)–(3) the coproduct in $\mathrm{U}_{\alpha,\beta}(h(1))$ takes the form

$$\Delta X = \mathrm{e}^{H'\cdot\alpha''} X'' + \mathrm{e}^{\beta'\cdot H''} X' \tag{11}$$

$$\Delta H = H' + H''. \tag{12}$$

Suppose that only the composition (10) is deformed

$$[X_1, X_2] = H + \Phi_{1,2}(\alpha, \beta; H).$$

Imposing the conditions $\Delta([X_1, X_2]) = [\Delta X_1, \Delta X_2]$ and $\Delta\Phi_{1,2}(\alpha, \beta; H) = \Phi_{1,2}(\alpha, \beta; H' + H'')$ one easily obtains the relation

$$\begin{aligned} &\mathrm{e}^{H' \operatorname{tr} \alpha}(H'' + \Phi_{1,2}(\alpha, \beta; H'')) + \mathrm{e}^{\operatorname{tr} \beta H''}(H' + \Phi_{1,2}(\alpha, \beta; H')) \\ &\qquad = H' + H'' + \Phi_{1,2}(\alpha, \beta; H' + H''). \end{aligned}$$

The solution is

$$\Phi_{1,2}(\alpha, \beta; H) = \frac{\mathrm{e}^{(\operatorname{tr} \alpha)H} - \mathrm{e}^{(\operatorname{tr} \beta)H}}{\operatorname{tr} \alpha - \operatorname{tr} \beta} - H.$$

Finally the deformed product takes the form

$$[X_1, X_2] = \frac{\mathrm{e}^{(\operatorname{tr} \alpha)H} - \mathrm{e}^{(\operatorname{tr} \beta)H}}{\operatorname{tr} \alpha - \operatorname{tr} \beta} \qquad [H, X_m] = 0. \tag{13}$$

Formulae (4)–(7) show that here the antipode and the co-unit remain trivial

$$S(H) = -H \qquad S(X_m) = -X_m \qquad S(1) = 1 \tag{14}$$

$$\varepsilon(H) = \varepsilon(X_m) = 0 \qquad \varepsilon(1) = 1. \tag{15}$$

The constructed quantum algebra $\mathrm{U}_{\alpha,\beta}(h(1))$ defined by the relations (11)–(15) is the five-parameter family. The properties of its members may be quite different. One can choose α and β proportional to the unit matrices, $\alpha = q \cdot I, \beta = p \cdot I$ and obtain some sort of the multiparametric deformation of $h(1)$. If $p = -q$ one gets the initial form of the quantized Heisenberg algebra proposed in [6]. While for nilpotent α and triangular β,

$$\alpha = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mu & \nu \\ 0 & \mu \end{pmatrix}$$

we have

$$\begin{aligned} &\Delta X_1 = X_1'' + \lambda H' X_2'' + (X_1' + \nu X_2' H'')\,\mathrm{e}^{\mu H''} \\ &\Delta X_2 = X_2'' + \mathrm{e}^{\mu H''} X_2' \\ &[X_1, X_2] = \mathrm{e}^{2\mu H}/2\mu. \end{aligned}$$

When μ tends to zero we re-obtain the initial form (10) of multiplication while the coproduct of X_1 remains modified.

Example 2. Let us construct the special type of quantization for the algebra of flat motions e(2):

$$[M, E_+] = E_+ \qquad [M, E_-] = -E_- \tag{16}$$

$$[E_+, E_-] = 0. \tag{17}$$

Here we choose $H = E_-, X_1 = E_+, X_2 = M$. Let $\alpha = \lambda \cdot I = -\beta$. Then the coproduct becomes

$$\begin{aligned} &\Delta E_+ = (\mathrm{e}^{\lambda E_-'})E_+'' + E_+'(\mathrm{e}^{-\lambda E_-''}) \\ &\Delta E_- = E_-' + E_-'' \\ &\Delta M = (\mathrm{e}^{\lambda E_-'})M'' + M'(\mathrm{e}^{-\lambda E_-''}). \end{aligned}$$

Let the commutator (17) remain stable and write the quantization of the composition (16) in the form

$$[M, E_+] = E_+\Phi_+(\lambda, E_-)$$

$$[M, E_-] = \Phi_-(\lambda, E_-).$$

The functions $\Phi_-(\lambda, E)$ and $\Phi_+(\lambda, E)$ have different limits:

$$\lim_{\lambda\to 0} \Phi_+(\lambda, E_-) = 1$$

$$\lim_{\lambda\to 0} \Phi_-(\lambda, E_-) = -E_-.$$

First consider the relation $\Delta[M, E_-] = [\Delta M, \Delta E_-]$. It gives the very simple equation

$$\Phi_-(\lambda, E'_-)\, e^{-\lambda E''_-} + e^{\lambda E'_-}\Phi_-(\lambda, E''_-) = \Phi_-(\lambda, E'_- + E''_-)$$

which has the obvious solution

$$\Phi_-(\lambda, E_-) = -\frac{1}{2\lambda}\,(e^{\lambda E_-} - e^{-\lambda E_-}).$$

The corresponding equation for $\Phi_+(\lambda, E_-)$ is

$$\begin{aligned}-\lambda\Phi_-(\lambda, E''_-)E'_+ + \lambda E''_+\Phi_-(\lambda, E'_-) + e^{\lambda(E'_- + E''_-)}E''_+\Phi_+(\lambda, E''_-)\\ + E'_+\Phi_+(\lambda, E'_-)\, e^{-\lambda(E'_- + E''_-)}\\ = e^{\lambda E''_-}E'_+\Phi_+(\lambda, E'_- + E''_-) + e^{-\lambda E'_-}E'_+\Phi_+(\lambda, E'_- + E''_-).\end{aligned}$$

Substituting the explicit form of $\Phi_-(\lambda, E)$ one obtains the result

$$\Phi_+(\lambda, E_-) = \tfrac{1}{2}(e^{\lambda E_-} + e^{-\lambda E_-})$$

which defines the final form of the quantized product

$$[M, E_+] = E_+ \cosh(\lambda E_-) \qquad [M, E_-] = -(1/\lambda)\sinh(\lambda E_-).$$

Here the deformed antipode (6) is also non-trivial:

$$S(M) = -e^{-\lambda E_-}M\, e^{\lambda E_-} = -M + \sinh(\lambda E_-)$$

$$S(E_-) = -E_- \qquad S(E_+) = -E_+.$$

This example shows that the proposed method of quantization works even when $\mathscr{G}$ and the subalgebra generated by $\{H_i\}$ do not form a reductive pair.

The standard Drinfeld–Jimbo quantization of simple Lie algebras can be incorporated in this scheme. The generators H_i, X_m can be treated as the Chevalley basis of simple Lie algebra. Matrices α^i, β^j must be taken in the form

$$(\alpha^i)^k_r = \delta^{ik}\delta^k_r q^i \qquad \beta^j = -\alpha^j \qquad q^i = q^{(\lambda_i, \lambda_i)/2}$$

where λ_i are the basic roots. Surely, in this approach, after the construction of the deformation functions (8) the Jacobi identity must be explicitly checked. In a forthcoming paper we shall show how to treat this situation systematically.

We thank Professor R Schrader and his collaborators for helpful discussions. One of us (VL) acknowledges the support and warm hospitality of the Freie Universität Berlin while finishing this work.

References

[1] Drinfeld V G 1986 Quantum groups *Proc. of the International Congress of Mathematicians* vol 1 (Berkeley, CA: Academic) pp 798–820

[2] Jimbo M 1986 *Lett. Math. Phys.* **11** 247

[3] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1989 *Alg. Anal.* **1** 178

[4] Kulish P P and Reshetikhin N Yu 1981 *Zap. Nauch. Semin. LOMI* **101** 101

[5] Demidov E E, Manin Yu I, Mukhin E E and Zhdanovich D V 1990 Non-standard quantum deformations of GL(n) and consistent solution of Y–B equations *Preprint* RIMS-701 Kyoto

[6] Celegini E, Giachetti R, Sorace E and Tarlini M 1990 *J. Math. Phys.* **31** 2548; 1991 *J. Math. Phys.* **32** 1155

[7] Celegini E, Giachetti R, Kulish P P, Sorace E and Tarlini M 1991 *Preprint* DFF 139/6/91 Firenze

[8] Kulish P P 1991 *Teor. Matem. Fizika* **86** 157

[9] Abe E 1980 *Hopf Algebras (Cambridge Tracts in Mathematics 74)* (Cambridge: Cambridge University Press)